MEC2120
Kinematics of Machines

Chapter 3:
Kinematic Synthesis of Planar Mechanisms

Kinematic Analysis vs. Kinematic Synthesis

- Kinematic analysis is the determination of the motion inherent in a given machine or mechanism.
- Formerly displacement analysis was of paramount interest, and it still may be.
- However, increases in rotational speeds have made a knowledge of velocity and acceleration characteristics critical factors in the design of the many elements comprising the complete machine.
- Analysis: Given a linkage, find the motion of its links, for a prescribed motion of its input joint(s).
- Kinematic synthesis is the reverse problem: it is the determination of mechanisms that are to fulfill certain motion specifications.
- Synthesis is the very ·fundamental of design, for it represents the creation of new hardware to meet particular needs in motion-displacement, velocity, or acceleration-singly or in combination.

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Kinematic Analysis vs. Kinematic Synthesis

Kinematic Synthesis: Given a task to be produced by a linkage, find the linkage that best performs the task.

The task at hand can be one of three, in this context:

(a) Function generation: the motion of the output joint(s) is prescribed as a function of the motion variable(s) of the input joint(s);

(b) Motion generation (a.k.a. rigid-body guidance): the motion of the output link(s) is prescribed in terms of the motion variable(s) of the input link(s) or joint(s);

(c) Path generation: the path traced by a point on a floating link—a link not anchored to the mechanism frame—is prescribed as a curve, possibly timed with the motion of the input joint(s).

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The overall problem of synthesis may be approached in three sometimes interrelated phases. It is necessary to reach decisions on

(I) the form or type of mechanism,

(II) the number of links and the nature of the connections needed to permit the required movability, and

(III) the proportions (lengths) of the links necessary to accomplish the specified motion transformation.

- The first phase is called **type synthesis.** Here the choice of the kind of links or constructional units is determined, as linkwork, gears, cams, belts, etc.
- The second phase, called **number synthesis,** deals with the number of links and the number of pairs of a given type required to obtain a given number of degrees of freedom, i.e., a given number of independent inputs to the mechanism.
- The third phase is called **dimensional synthesis.**

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Both type and number synthesis come under one single umbrella called **Qualitative synthesis**

Qualitative synthesis

Type synthesis: Given a task to be produced by a mechanism, find the type that will best perform it, e.g., a linkage, a cam mechanism, a gear train, or a combination thereof.

Number synthesis: Given a task to be produced by a mechanism of a given type, find the number of links and joints that will best execute the task.

- Both type and number syntheses pertain to the conceptual design phase, as the former refers to choosing the type of mechanism to perform the required function, namely, a linkage, a cam-follower mechanism, a belt-pulley transmission, or a gear train, for example.
- Number synthesis refers to the numbers of links and joints in a linkage, along with the type of joints to be used—kinematic joints, or lower kinematic pairs

- Qualitative synthesis focuses on the synthesis of linkages.

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Number Synthesis ⇒ It refers to the determination of the number and types (order ie binary, ternary etc.) of various links and the no. of simple pairs which give rise to single DOF planar linkages.

From Kutzbach equ.

$$F = 3(n-1) - 2j$$

for $F = 1$, $3n - 2j - 4 = 0 \Rightarrow 3n = 2j + 4$.

which means total no. of links must be even

Further, the min. no. of binary links is 4

(i) Highest order of a link in (n) link mechanism is

$$i = \frac{n}{2}$$

So for a 6 link mechanism

$$i = 3$$

$n_2 + n_3 = 6$ Since min. $n_2 = 4$, $n_3 = 2$ Further $3n = 2j + 4$ so, $j = 7$

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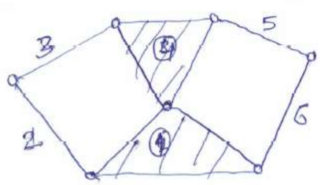
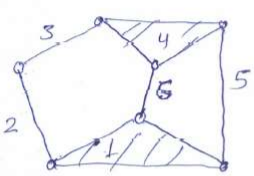
Thus a 6-link mechanism must have 4 binary links and 2 ternary links connected by 7 simple pairs.

Two different arrangement are possible



Watts chain

Two ternary links directly joined



Stephenson's Chain

Two ternary links connected by binary link

If $n = 8$; $i = \frac{n}{2} = 4$

$n_2 + n_3 + n_4 = 8$ & $n_2 = 4 + n_4$

$n_2 = 4$; $n_3 = 4$; $n_4 = 0$

$n_2 = 5$; $n_3 = 2$, $n_4 = 1$

$n_2 = 6$; $n_3 = 0$, $n_4 = 2$

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- As both type and number synthesis come under one single umbrella called **Qualitative synthesis.**
- For consistency, dimensional synthesis is termed **Quantitative synthesis**.

Quantitative synthesis (i.e. Dimensional synthesis)

- In dimensional synthesis the main objective is to find the dimensions defining the geometry of the various links and joints of the kinematic chain underlying the mechanism under design.
- Dimensioning involves two phases: functional dimensioning and mechanical dimensioning.
- The former is previous to the latter, and includes the determination of the fundamental dimensions of the machine, prior to the shaping of all its parts.
- It is the functional dimensioning where kinematic synthesis plays a major role.
- Mechanical dimensioning pertains to the dimensioning of the machine elements for stress, strength, heat capacity, and dynamic-response requirements.

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Quantitative synthesis (Dimensional synthesis)

There are, moreover, two types of dimensional synthesis:

1. Exact synthesis: Number of linkage parameters available is sufficient to produce exactly the prescribed motion. Problem leads to—linear or, most frequently, nonlinear—equation solving.

2. Approximate synthesis: Number of linkage parameters available is not sufficient to produce exactly the prescribed motion. Optimum dimensions are sought that approximate the prescribed motion with the minimum error. Problem leads to mathematical programming (optimization)

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Quantitative synthesis (Dimensional synthesis)

Furthermore, quantitative synthesis can be achieved, with a variable degree of success, via one of three types of methods:

Graphical:

•Under this type, the geometric relations of the task at hand are manipulated directly as such.

•In the pre-computer era this was done by means of drafting instruments alone.

•Nowadays, the drafting instruments have been replaced by CAD software.

Algebraic:

•In these methods, the geometric relations in question are manipulated by algebraic means of computer-algebra software, to produce the desired linkage parameters as the solutions to the underlying synthesis equations.

•The geometric relations of any linkage containing any combination of five of the six lower kinematic pairs, the screw pair excluded, lead to systems of multivariate polynomial equations.

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Quantitative synthesis (Dimensional synthesis)

•Most of the computer-algebra software available caters to systems of multivariate polynomial equations.

Semigraphical:

- Purely algebraic methods entail some drawbacks, like algebraic singularities, which are conditions under which some solutions cannot be found for reasons other than kinematic.
- Semigraphical methods reduce the system of algebraic equations to a subsystem of bivariate equations, i.e., equations involving only two variables.
- The bivariate equations defining a set of contours in the plane of those two variables, the real solutions to the problem at hand are found as the intersections of all those contours.

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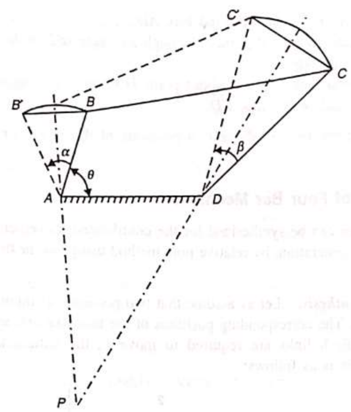
Synthesis by Relative Pole Method

The pole is a centre of rotation of moving link relative to a fixed link of the mechanism. To locate the pole point, let us consider a four bar mechanism *ABCD* in which link *AD* is fixed and crank *AB* is at initial rotation angle θ. If the crank *AB* is rotated by angle α in anti-clockwise direction to take new position *AB′*, the follower link *CD* rotates by angle β to reorient the coupler link *BC* to the new position *B′C′* as shown in Figure . The normals drawn at the mid-points of *BB′* and *CC′* intersect at point *P*. This point is centre of rotation of coupler relative to the fixed link *AD* and is called **pole.**



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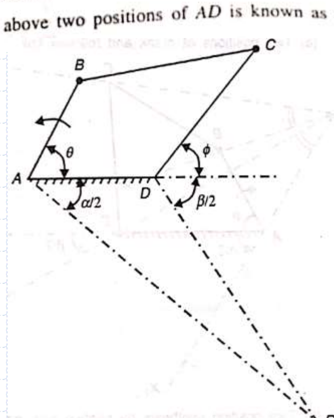
Synthesis by Relative Pole Method

A relative pole is a centre of rotation of a link relative to other moving links such as crank and follower. In a four bar mechanism, if the crank and the follower links rotate by angles α and β respectively from their initial positions θ and ϕ, then the relative pole of coupler link relative to crank and follower can be found by the following procedure :

1. Join nodes *A* and *D* of the fixed link *AD*.
2. Rotate the link *AD* about point *A* through an angle $\alpha/2$ in the direction opposite to that of the link *AB*.
3. Similarly, rotate the link *AD* about point *D* through an angle $\beta/2$ in the direction opposite to that of the link *CD*.

The point of intersection of above two positions of *AD* is known as relative pole and is represented by *R*.



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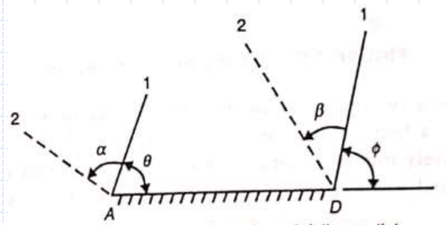
Synthesis of Four Bar Mechanism using Rlative Pole Method

Two position synthesis. Let us assume that two positions of input link crank separated by angle α are given. The corresponding positions of the follower link are separated by angle β [Figure]. Both links are required to move in the anticlockwise direction. The procedure of synthesis is as follows:



(a) Two positions of crank and follower link

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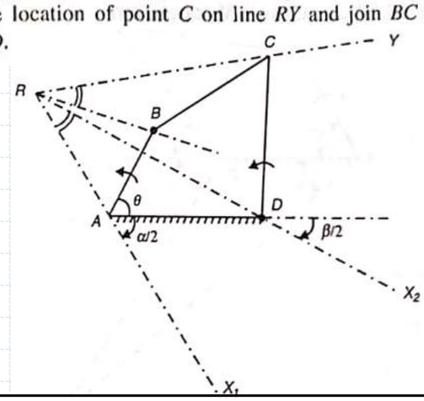
Synthesis of Four Bar Mechanism using Rlative Pole Method
Two Position Synthesis

1. Choose the frame AD arbitrarily at certain distance apart [See Figure].
2. Draw a line AX_1 passing through point A and inclined at an angle $\alpha/2$ from link AD in the direction opposite to the direction of rotation of the link AB.
3. Similarly, draw another line DX_2 passing through point D inclined at angle $\beta/2$ from link AD in the direction opposite to the direction of rotation of link CD.
4. The intersection of lines AX_1 and DX_2 at point R is known as **relative pole.**
5. Choose suitable length for crank AB and mark the first position of the crank.
6. Join the points B and R by a straight line.
7. Draw a line RY from point R such that angle ARD is equal to angle BRY, i.e. the angle subtended by the frame is equal to the angle subtended by the coupler link.
8. Select the suitable location of point C on line RY and join BC and CD to form a mechanism $ABCD$.



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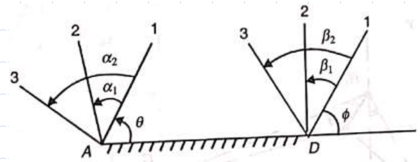
Synthesis of Four Bar Mechanism using Rlative Pole Method

Three position Synthesis

Three position synthesis. In three position synthesis, two pairs of the crank and follower rotations are required to be coordinated. Let the two pairs of coordinates indicating three positions of crank and follower are (α_1, β_1) and (α_2, β_2) respectively [See Figure].



(a) Three positions of crank and follower

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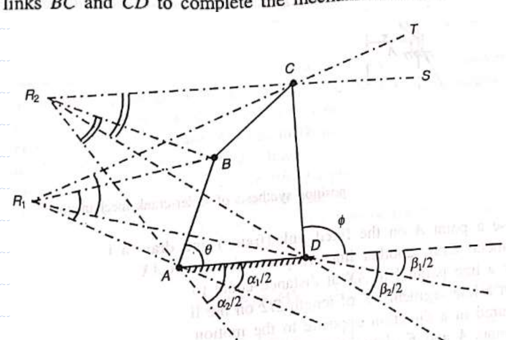
Synthesis of Four Bar Mechanism using Rlative Pole Method

Three position Synthesis

The stepwise procedure is as given below (refer Figure).

1. Choose the frame AD arbitrarily at certain distance apart.
2. Locate the relative poles R_1 and R_2 as described previously for two pairs of coordinates (α_1, β_1) and (α_2, β_2).
3. Choose suitable length of crank AB and mark the first position of the crank.
4. As the crank and follower subtend equal angles at the relative pole, so draw a line R_1T such that angle DR_1T is equal to angle AR_1B.
5. Join R_2 and B by a straight line.
6. Draw a line R_2S such that angle AR_2B is equal to angle DR_2S.
7. Locate the point C at the intersection of lines R_1T and R_2S.
8. Join links BC and CD to complete the mechanism $ABCD$.



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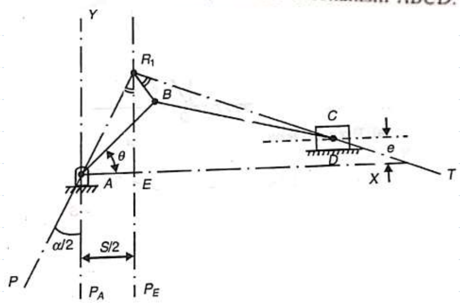
Synthesis of Slider Crank Mechanism using Rlative Pole Method
Two Position Synthesis

1. Choose a point A on the fixed link (frame) and draw a line AX along the slider movement. Draw another line AY perpendicular to AX.
2. Draw a line parallel to AX at distance equal to the required offset distance e.
3. Select a line segment AE of length $S/2$ on the line AX such that the distance AE is measured in a direction opposite to the motion of the slider.
4. At points A and E, draw perpendicular lines P_A and P_E.
5. Draw a line PR_1 at an angle $\alpha/2$ with line P_A in the direction opposite to the rotation of crank.
6. The intersection of line PR_1 with line P_E is relative pole R_1.
7. Choose suitable length of crank AB and mark its first position.
8. Join B with R_1 by a straight line.
9. Construct angle BR_1T equals to angle AR_1E. The line R_1T intersects the offset line at point C. Join BC to complete the mechanism $ABCD$.



Consider a slider crank mechanism in which when crank takes turn by angle α in anticlockwise direction from the first position, the slider changes its position by a distance S.

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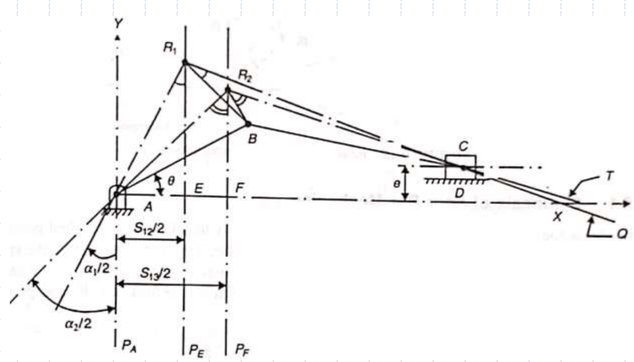
Synthesis of Slider Crank Mechanism using Rlative Pole Method
Three Position Synthesis

1. Choose point A on the fixed link and draw a line AX along the movement of slider.
2. Locate the relative poles R_1 and R_2 for two pairs of coordinates (α_1, S_{12}) and (α_2, S_{13}) respectively (as described in two position synthesis method).
3. Choose the length of crank AB and mark its first position.
4. Join B with R_1 and R_2 by straight lines.
5. Draw a line R_1Q making an angle BR_1Q equals to angle AR_1E.
6. Similarly, draw another line R_2T which makes an angle BR_2T equals to angle AR_2F.
7. Mark the position of slider C at the intersection of these two lines.
8. Join BC to form a slider-crank mechanism.



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Synthesis of Mechanism by Analytical Method

- The movement of the driver or input link can be represented by the input variable 'x' and that of the driven link or follower by the dependent variable 'y'.
- Both of them may have some prescribed functional relationship, say $y = f(x)$.
- There are two approaches : approximate synthesis and exact synthesis.
- By approximate synthesis, we mean that the function generated by the mechanism fits the defined function only at a finite number of points in the interval and intersects the desired path at a finite number of points.
- By exact synthesis, we mean that the generated function or path fits the desired function or path at all points in the interval.
- Exact synthesis is limited to a few arbitrary functions, on the other hand approximate synthesis pertains to almost all functions.
- The points at which the generated and desired functions meet are called precision points or accuracy points.
- The number of such points is equal to the number of design parameters at our disposal. This number varies from three to six.
- The computation difficulties increase with increase in the number of accuracy points.

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Synthesis of Mechanism by Analytical Method

Chebyshev's Spacing of Accuracy Points

- Let x_i and x_f be initial and final values respectively of variable x.
- The function $f(x)$ is desired to be generated in the interval x_i to x_f.
- Let the generated function be $F(x, R_1, R_2, R_3, \ldots R_k)$where $R_1, R_2, R_3, \ldots R_k$be design parameters.
- The difference $E(x)$ between the desired function and generated function can be represented by

$$E(x) = f(x) - F(x, R_1, R_2, R_3, \ldots R_k)$$

- At precision points or accuracy points say for $x = x_1, x_2, x_3, \ldots x_k$ desired and generated functions agree and $E(x)$ will be zero.
- At other points $E(x)$ will have some value and that will be the error. $E(x)$ is called 'structural error'.
- It is desirable that $E(x)$ should be minimum and therefore, spacing of accuracy points is very important.
- An exact analysis for $E(x)$ to be minimum is extremely difficult. The Chebyshev's spacing of accuracy points can always be taken as a first approximation.

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Synthesis of Mechanism by Analytical Method

Chebyshev's Spacing of Accuracy Points

- Chebyshev noted that the best linkage approximation of a given mechanism to any function occurs when the **absolute value of the maximum structural error between the precision points and both ends of the range are equalized**. Chebyshev's spacing of precision points is employed to minimize the structural error
- The accuracy points according to Chebyshev's spacing are given by

$$x_j = a - h\cos\frac{(2j-1)\pi}{2n}$$

Where $a = \frac{(x_i + x_f)}{2}$ and h= $\frac{(x_f - x_i)}{2}$

j = jth number of spacing point.
x_i: initial value of input variable
x_f: final value of input variable
n: number of spacing points

- The accuracy points can also be obtained by the **graphical method** as explained in the following steps :

(a) Draw a circle of radius 'h' and centre on the x-axis at a distance 'a' from O.
(b) Inscribe a regular polygon of side '$2n$' in this circle such that the two sides are perpendicular to the x-axis.
(c) Determine the locations of 'n' accuracy points by projecting the vertices on x-axis.

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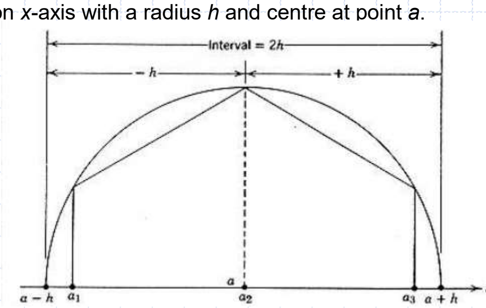
Synthesis of Mechanism by Analytical Method

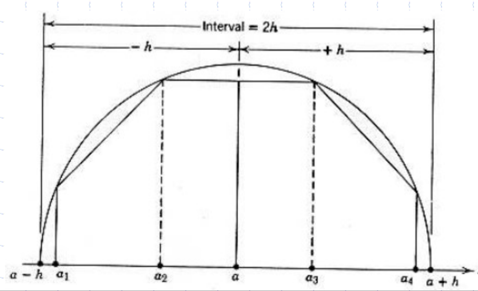
Chebyshev's Spacing of Accuracy Points

A semi-circle is drawn on x-axis with a radius h and centre at point a.

Three-accuracy points:



Four-accuracy points:



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Synthesis of Mechanism by Analytical Method

Chebyshev's Spacing of Accuracy Points

Problem: Find the three accuracy points using Chebyshev's Spacing, if a four bar mechanism is to generate the function $y = \frac{1}{x^2}$ over the range $1 \leq x \leq 2$.

Let the minimum and maximum values of independent variable x be called x_i and x_f . Our three precision points x_1, x_2, x_3 will fit between x_i and x_f; the sequence will be $[x_i, x_1, x_2, x_3, x_f]$.

For 3 points

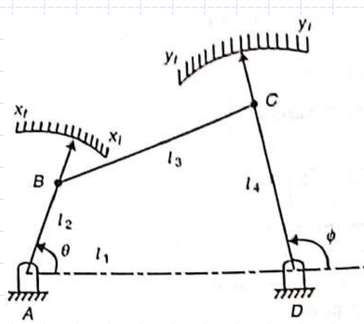
$$x_j = \frac{(x_i + x_f)}{2} - \frac{(x_f - x_i)}{2} \cos \frac{(2j-1)\pi}{6}, j = 1,2,3.$$

The three Chebyshev precision points obtained are:

$$x_1 = 1.0670$$
$$x_2 = 1.5000$$
$$x_3 = 1.9330$$

Function Generation by Mechanisms

Let θ and $\emptyset$ be the angles of rotation of input or driver link and driven or follower link respectively.
Let $y = f(x)$ be the function to be generated. The angle of rotation of the input or driver link *AB* represents independent variable x. The angle of rotation $\emptyset$ of the follower link *DC* represents dependent variable y as indicated in Figure. The relation between x and θ and that between y and $\emptyset$ is usually assumed to be linear. Let θi and $\emptyset_i$ be the initial values of θ and $\emptyset$ representing x_i and y_i respectively.



Function Generation by Mechanisms

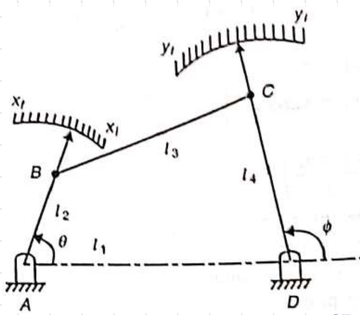
On the basis of an assumed linear relationship, we have

$$\frac{\theta - \theta_i}{x - x_i} = \text{constant} = r_x \text{ (say)} = \frac{\theta_f - \theta_i}{x_f - x_i}$$

Similarly,

$$\frac{\phi - \phi_i}{y - y_i} = \text{constant} = r_y \text{ (say)} = \frac{\phi_f - \phi_i}{y_f - y_i}$$

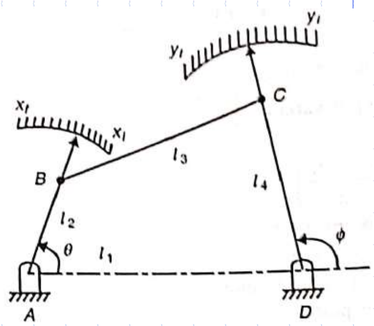
where r_x and r_y are scale factors. The subscripts *i* and *f* denote the initial and final values.



A four bar mechanism is to be designed, by using three precision points, to generate the function
1.5 y x = , for the range 1x4 ≤ ≤ .
Assuming 30° starting position and 120° finishing position for the input link and 90° starting position and 180° finishing position for the output link, find the values of x, y, θ and φ corresponding to the three precision points.

Freudenstein's Equation for Four Bar Chain for Three Accuracy Points

There are seven design parameters.
There are θ, $\emptyset$, r_x and r_y and three ratios of link lengths $\frac{l_1}{l_2}, \frac{l_1}{l_3}, \frac{l_1}{l_4}$.
The parameters symbol θ, $\emptyset$, r_x and r_y are either given or assumed and the three length ratios $\frac{l_1}{l_2}, \frac{l_1}{l_3}, \frac{l_1}{l_4}$ are to be determined as design parameters.



The coordinates of the point *B* are ($l_2$cos θ, l_2 sin θ) and for point *C* are

$$\left((l_1 + l_4 \cos\theta), l_4 \sin\theta\right)$$

The length of coupler link l_3 can be determined as

$$BC^2 = l_3^2 = (l_2 \cos\theta - l_1 - l_4 \cos\phi)^2 + (l_2 \sin\theta - l_4 \sin\phi)^2$$

or $$l_3^2 = l_1^2 + l_2^2 + l_4^2 + 2l_1l_4\cos\phi - 2l_1l_2\cos\theta - 2l_2l_4 \times (\cos\theta\cos\phi + \sin\theta\sin\phi)$$

or $$l_1^2 + l_2^2 - l_3^2 + l_4^2 + 2l_1l_4\cos\phi - 2l_1l_2\cos\theta = 2l_2l_4\cos(\theta - \phi)$$

Freudenstein's Equation for Four Bar Chain for Three Accuracy Points

$$l_1^2 + l_2^2 - l_3^2 + l_4^2 + 2l_1l_4\cos\phi - 2l_1l_2\cos\theta = 2l_2l_4\cos(\theta - \phi)$$

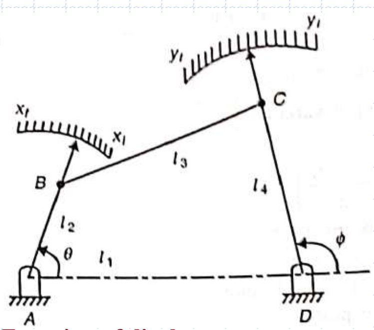
Dividing the above equation by $2l_2l_4$

$$\frac{l_1}{l_2}\cos\phi - \frac{l_1}{l_4}\cos\theta + \frac{l_1^2 + l_2^2 - l_3^2 + l_4^2}{2l_2l_4} = \cos(\theta - \phi)$$

or $$k_1\cos\phi - k_2\cos\theta + k_3 = \cos(\theta - \phi)$$

where

$$k_1 = \frac{l_1}{l_2}, \quad k_2 = \frac{l_1}{l_4} \quad \text{and} \quad k_3 = \frac{l_1^2 + l_2^2 - l_3^2 + l_4^2}{2l_2l_4}$$



The above Equation is known as **Freudenstein's Equation of displacement**

The following steps are followed to determine length ratios :
(a) Obtain three accuracy points by using Chebyshev's spacing, i.e. x^1, x^2 and x^3 from Eq. (7.10).
(b) Obtain the corresponding values of y^1, y^2 and y^3 by using given function $y = f(x)$.
(c) By using Eqs. (7.11) and (7.12), determine (θ^1, θ^2 and θ^3) and (φ^1, φ^2, and φ^3) corresponding to values of x and y.
(d) Obtain three equations in K^1, K^2 and K^3 by substituting the three related pairs of the angles (θ^1 and φ^1), (θ^2 and φ^2), and (θ^3 and φ^3).
(e) Solve these three equations for K^1, K^2 and K^3.
(f) Determine length ratios from values of K^1, K^2 and K^3.
(g) Assume one link length or it may be given. Determine other lengths which can be determined from the length ratios. The lengths l^1 and l^3 will always be positive. The negative values of l^2 or l^4 can be interpreted in the vector sense. The negative sense will mean that their layouts are opposite to the fixed link.

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Determine the lengths of all the four links in a four bar chain for the length of the smallest being 10 cm to generate $y = log_{10}x$ in the interval $1 \leq x \leq 10$ for three accuracy points. The range of angles of input link and output link are $45^o \leq \theta \leq 105^o$ and $135^o \leq \varphi \leq 225^o$.

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